

NUMERICAL SOLUTION OF A CAUCHY PROBLEM IN ANNULAR DOMAINS

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Abstract - We consider the Cauchy problem of recovering both Neumann and Dirichlet data on the inner part of the boundary of an annular domain, from measurements on some part of the outer boundary. Using tools from complex analysis and Hardy classes approximation, we provide a constructive and robust identification scheme, together with numerical experiments.

1. INTRODUCTION

1.1. Motivation

The problem we are dealing with in this contribution is to recover both Dirichlet and Neumann boundary data on the internal boundary of an annulus, or a Robin coefficient which is actually the quotient of these data, from measurements on some part of the outer boundary. Such a problem arises for example in corrosion detection in tubular domains, or in electroencephalography applications of recovering epilepsy centers in a brain. In such a case, the annular domain is derived from the head by a conformally mapping [9]. Such problems have been widely studied for simply connected domains, that can be conformally mapped on the unit disk [4]. The method we wish to generalize to annular domains is to construct harmonic approximations by solving a bounded extremal problem there. Such a construction uses an explicit asymptotic expansion of the harmonic approximant, together with the determination of the actual bound of that approximant, stabilizing the whole algorithm by a cross validation procedure.

To that end, the first issue to tackle is to get explicit asymptotic expansions in annular domains. Provided full data are available on the whole of the outer boundary, such formulae have been obtained in [6]. In the present work, we present a numerical algorithm making use of them has been set up. The regularization has been achieved by characterizing the actual bound as the unique zero of an appropriate function. The algorithm is proved to be robust with respect to noise. The first part of this contribution is devoted to presenting the case of full data on the outer boundary, including theoretical results (explicit formulae, stability and robustness), as well as numerical experiments. However, full data cannot be expected in several cases, and especially in electroencephalography applications. When data are lacking on some part of the external boundary, explicit formulae of the analytic approximant have therefore been sought and obtained, and used as a basic tool in the algorithmic part. Regularization can be obtained by the cross validation technique present in [5], needing to devote some part of the outer boundary data to that task, which can be handled by the explicit formulae. Characterizing the actual bound as the unique zero of an appropriate function, moreover quite easy to compute, turns out to be however a cheaper way to proceed, and most efforts have therefore been focused on it. We shall be presenting and comparing numerical results using both these methods.

1.2. Inverse problem

More precisely, let \mathbb{D} be the unit disc and G be the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed s with $0 < s < 1$ and denote $\partial G = \mathbb{T} \cup s\mathbb{T}$.

Let I be a non-null measurable subset of \mathbb{T} , and let $J = \partial G \setminus I$. We consider the following inverse problem: given functions u_d and Φ , or a number of pointwise measurements, with $\Phi \neq 0$, find a function q , such that a solution u to

$$\begin{cases} \Delta u = 0 & \text{in } G & (i) \\ u = u_d & \text{on } I & (ii) \\ \partial_n u = \Phi & \text{on } I & (iii) \end{cases} \quad (1)$$

also satisfies

$$\partial_n u + q u = 0 \text{ on } J, \quad (2)$$

where ∂_n stands for the partial derivative w.r.t. the outer normal unit vector to \mathbb{T} . In the thermal framework, u_d and Φ correspond to the measured temperature and to the imposed heat flux on the outer boundary of some plane section of a tube, while q is the exchange Robin coefficient to be recovered on the associated inner boundary.

Let $c, C > 0$ and introduce the following class of "admissible" Robin coefficients

$$A^n = \{q \in C^n(J), |q^{(k)}| \leq C, 0 \leq k \leq n, \text{ and } q \geq c\}$$

Theorem 1.1 [5] *Let $n \geq 0$, $\Phi \in W^{n,2}(I)$, $\Phi \geq 0$, $\Phi \neq 0$ and assume that $q \in A^n$ for some constants $c, C > 0$. Then there exists a unique function $u \in W^{n+3/2,2}(G)$, whence $u_d \in W^{n+1,2}(\partial G)$, solution to (1)_i, (1)_{iii} and (2). Further, there exist constants $m > 0$ and κ (depending on the class A^n) such that for all $q \in A^n$ and $\Phi \in W^{n,2}(I)$,*

$$u \geq m > 0 \text{ on } J,$$

and

$$\|u\|_{W^{n+1,2}(\partial G)} \leq \kappa.$$

The proofs of the above results rely on shift and Sobolev imbeddings Theorems, together with the Hopf maximum principle.

The next identifiability property ensures the uniqueness of solutions q to the inverse problem, which is a necessary prerequisite for stability issues to make sense.

Theorem 1.2 *Let $\Phi \in L^2(I)$, $\Phi \geq 0$, $\Phi \neq 0$ and $q_1, q_2 \in A^0$. Let u_1 and u_2 be the associated solutions of (1)_i, (1)_{iii} and (2). If $u_{1|_I} = u_{2|_I}$, then $q_1 = q_2$.*

1.3. Harmonic conjugate

Let $\Phi \in L^2(I)$ and assume that $q \in A^0$. From Theorem 1.1, $u|_{\partial G} \in W^{1,2}(\partial G)$. Then there exists a function v harmonic in G such that $\partial_\theta v = \partial_n u$ on ∂G , where ∂_θ stands for the tangential partial derivative on ∂G , from Cauchy-Riemann equations. Hence, v is given on I up to a constant by

$$v|_I(e^{i\theta}) = \int_{\theta_0}^{\theta} \Phi(e^{i\tau}) d\tau.$$

Further, from the M. Riesz theorem [8, Thm 4.1], the harmonic conjugate operator is bounded in $L^2(\partial G)$, whence $v|_{\partial G} \in W^{1,2}(\partial G)$. Thus, $f = u + i v$ is analytic in G and $f|_{\partial G} \in W^{1,2}(\partial G)$; it is given on I by

$$f(e^{i\theta}) = u_d(e^{i\theta}) + i \int_{\theta_0}^{\theta} \Phi(e^{i\tau}) d\tau. \quad (3)$$

Also, on J ,

$$q = -\frac{\partial_\theta v}{u} = -\frac{\partial_\theta \text{Im} f}{\text{Re} f}, \quad (4)$$

which gives the link to be used between q and f , in order to recover q from approximations to f on I of the boundary ∂G .

2. BOUNDED EXTREMAL PROBLEM

2.1. Hardy classes of circular domains

Let G be a *circular domain*, that is, a domain consisting of the open unit disc from which a finite number of pairwise disjoint closed discs have been removed:

$$G = \mathbb{D} \setminus \bigcup_{j=1}^N (a_j + r_j \overline{\mathbb{D}}), \quad (5)$$

with the obvious inequalities satisfied by the a_j and r_j for $j = 1, \dots, N$. We write $D_j = a_j + r_j \mathbb{D}$ for $1 \leq j \leq N$. Let Γ denote the boundary of G . We normalize the Lebesgue measure on Γ so that each circle Γ_j composing it is given unit measure.

The Hardy spaces $H^p(G)$ on the domain G were defined by Rudin [10] in terms of analytic functions f such that $|f(z)|^p$ has a *harmonic majorant* on G , that is, a real harmonic function $u(z)$ such that $|f(z)|^p \leq u(z)$ on G . It is also possible to define the Hardy spaces $H^p(\partial G)$ for $1 \leq p < \infty$, as the closure in $L^p(\partial G)$ of the set R_G of rational functions whose poles lie in the complement of \overline{G} . This approach, similar to the one in [3], was taken in [7]. The spaces $H^p(G)$ and $H^p(\partial G)$ are then isomorphic in a natural way, and so we identify the two spaces.

Below, we stick to the most completely analysed example of the annulus $G = \mathbb{D} \setminus s\mathbb{D}$ for some fixed s with $0 < s < 1$ and to the Hilbert case $p = 2$. Here, the Lebesgue measure on ∂G is normalized so that the circles \mathbb{T} and $s\mathbb{T}$ each have unit measure.

The space $H^2(\partial G)$ has a canonical orthonormal basis consisting of the functions

$$e_n(z) := (z^n / \sqrt{1 + s^{2n}})_{n \in \mathbb{Z}},$$

and it can be written as an orthogonal direct sum

$$H^2(\partial G) = H^2(\mathbb{D}) \oplus H_0^2(\mathbb{C} \setminus s\overline{\mathbb{D}})$$

of elementary Hardy spaces, by taking the closed linear spans of $(e_n)_{n \geq 0}$ and $(e_n)_{n < 0}$ respectively. Here $H_0^2(\mathbb{C} \setminus s\overline{\mathbb{D}})$ is the Hardy space of functions analytic on the complement of $s\overline{\mathbb{D}}$, with L^2 boundary values, and vanishing at infinity. It should be noted that a similar decomposition applies to general spaces $H^p(\partial G)$, but the direct sum is no longer orthogonal in the case $p \neq 2$.

So, a function $f \in H^2(G)$ has the following expansion:

$$f(z) = \sum_{n \in \mathbb{Z}} a_n z^n, \quad \text{where } \|f\|^2 = \sum_{n \in \mathbb{Z}} (1 + s^{2n}) |a_n|^2 \text{ for } z \in G.$$

We write $P_{L^2(I)} g = \chi_I g$ for the function in $L^2(\partial G)$ that coincides with g on I and vanishes on J . The definition of $P_{L^2(J)}$ is analogous.

2.2. Approximation in Hardy classes

We assume that $I = [-\theta_0, \theta_0] \subset \mathbb{T}$, $0 < \theta_0 < \pi$. We write $L^2(\partial G) = L^2(I) \oplus L^2(J)$. We suppose that we are given $f \in L^2(I)$ and we wish to approximate f as well as possible by the restriction to I of an $H^2(\partial G)$ function i.e. $P_{L^2(I)} g$ for $g \in H^2(\partial G)$. In view of the results established in [7], the space $P_{L^2(I)} H^2(\partial G)$ is dense in $L^2(I)$. Then there will exist a sequence (g_n) of $H^2(\partial G)$ functions such that $\|P_{L^2(I)} g_n - f\|_{L^2(I)} \rightarrow 0$. However, if $f \neq P_{L^2(I)} g$ for any $g \in H^2(\partial G)$ then it will follow that $\|P_{L^2(J)} g_n\|_{L^2(J)} \rightarrow \infty$, i.e. the approximation problem is ill-posed.

This motivates the following bounded extremal problem **BEP**.

Problem 2.1 Let $f \in L^2(I) \setminus P_{L^2(I)} H^2(\partial G)$, $f_1 \in L^2(J)$ and $M > 0$. Find a function $g \in H^2(\partial G)$ such that $\|g - f_1\|_{L^2(J)} \leq M$ and

$$\|f - g\|_{L^2(I)} = \inf\{\|f - \psi\|_{L^2(I)} : \psi \in H^2(\partial G), \|\psi - f_1\|_{L^2(J)} \leq M\}. \quad (6)$$

We shall require the solution to the **BEP** for $H^2(\partial G)$, which can be expressed as follows. If \mathcal{T} is the Toeplitz like operator on $H^2(\partial G)$ defined by $\mathcal{T}g = P_{H^2(\partial G)} P_{L^2(J)} g = P_{H^2(\partial G)} \chi_J g$, then the solution g of Problem 2.1 is given by the formula

$$(1 + \lambda \mathcal{T})g = P_{H^2(\partial G)} [f + (1 + \lambda) f_1],$$

for the unique $\lambda > -1$ such that $\|g - f_1\|_{L^2(J)} = M$.

Notice that the map $\chi \rightarrow \mathcal{T}_\chi$ for $\chi \in L^\infty(\partial G)$ is linear and contractive. It is, in fact, isometric. If the function $\chi = \chi_J$ is the characteristic function of the component $J = s\mathbb{T}$ of the boundary of G , then \mathcal{T}_{χ_J} is self-adjoint, $\sigma_e(\mathcal{T}_{\chi_J}) = \{0, 1\}$, and 0 and 1 are not eigenvalues of \mathcal{T}_{χ_J} . Thus, the spectrum of \mathcal{T}_{χ_J} consists of $\{0, 1\}$ and eigenvalues of finite multiplicity in $(0, 1)$ which accumulate at 0 and 1. Hence, the spectrum of \mathcal{T}_{χ_J} is disconnected, in fact, totally disconnected (see [1, 11]).

We may now apply the theory above to obtain the solution to the **BEP** for $H^2(\partial G)$, which can be expressed as follows.

Whenever κ_1 is defined on I and κ_2 on J , we write $\kappa_1 \vee \kappa_2$ for the function equal to κ_1 on I and κ_2 on J .

It can be checked that, if we denote by ϕ the function in $L^2(\partial G)$ defined by

$$\phi := f \vee 0 + 0 \vee (1 + \lambda) f_1,$$

then we have

Lemma 2.1

$$P_{H^2(\partial G)} \phi(z) = \sum_{n \in \mathbb{Z}} \frac{a_n + b_n s^{2n}}{1 + s^{2n}} z^n,$$

where

$$a_n = \frac{1}{2\pi} \left(\int_{-\theta_0}^{\theta_0} f(e^{i\theta}) e^{-in\theta} d\theta + (1 + \lambda) \int_{\theta_0}^{2\pi - \theta_0} f_1(e^{i\theta}) e^{-in\theta} d\theta \right),$$

and

$$b_n s^n = \frac{1 + \lambda}{2\pi} \int_0^{2\pi} f_1(se^{i\theta}) e^{-in\theta} d\theta.$$

Lemma 2.2 Let $g \in H^2(\partial G)$ such that $g(z) = \sum_{n \in \mathbb{Z}} g_n z^n$ for $z \in G$ and \mathcal{T} the Toeplitz operator. Then

$$\mathcal{T}g(z) = \sum_{n \in \mathbb{Z}} \frac{1}{1 + s^{2n}} \left(g_n \left(1 + s^{2n} - \frac{\theta_0}{\pi} \right) - \sum_{m \neq n} g_m \frac{\sin(m - n)\theta_0}{\pi(m - n)} \right) z^n. \tag{7}$$

Then one can calculate the solution g of Problem 2.1 by solving the following system:

$$(1 + \lambda \mathcal{T})g = \mathcal{B}.$$

On the Fourier basis, the operator \mathcal{T} is a semi-infinite Toeplitz matrix: for $n, m \in \mathbb{Z}$

$$\mathcal{T}_{n,m} = \begin{cases} \frac{1}{1 + s^{2n}} \left(1 + s^{2n} - \frac{\theta_0}{\pi} \right) & \text{when } n = m \\ -\frac{1}{1 + s^{2n}} \frac{\sin(m - n)\theta_0}{\pi(m - n)} & \text{when } n \neq m, \end{cases}$$

and

$$\mathcal{B}_n = \frac{a_n + b_n s^{2n}}{1 + s^{2n}}.$$

The behavior with respect to λ of the error $e(\lambda)$ and of the constraint $M(\lambda)$ defined by:

$$e(\lambda) = \|f - g(\lambda)\|_{L^2(I)}, \quad M(\lambda) = \|g(\lambda) - f_1\|_{L^2(J)},$$

is smooth and monotonic. In particular, as $\lambda \searrow -1$,

$$e(\lambda) \searrow 0, \quad M(\lambda) \nearrow \infty, \quad \text{if } f \notin H^2(\partial G), \quad M(\lambda) \nearrow \|f - f_1\|_{L^2(J)}, \quad \text{if } f \in H^2(\partial G).$$

These are simple generalizations of results found in [2, Section 3].

From now on, and for simplicity, we choose $f_1 = 0$.

2.3. Continuity of the solutions with respect to the data

In this Section, we shall be concerned with continuity properties of the solutions of Problem 2.1 with respect to the data f and M . Let Ψ be the mapping defined by:

$$\begin{aligned} \Psi &: L^2(I) \times \mathbb{R}_+^* \rightarrow H^2(\partial G) \\ (f, M) &\mapsto g(f, M), \end{aligned}$$

with $g(f, M)$ is the solution of Problem 2.1 associate to the data f and M . Let $\mathcal{D} = \{(h, M) \in H^2(\partial G)|_I \times \mathbb{R}_+^* \mid \|h\|_{L^2(J)} < M\}$.

Theorem 2.1 *The mapping Ψ is continuous on the whole $L^2(I) \times \mathbb{R}_+^*$ with respect to the weak topology of $H^2(\partial G)$, whereas it is only continuous on $(L^2(I) \times \mathbb{R}_+^*) \setminus \mathcal{D}$, with respect to its strong topology.*

Steps of proof:

We define the mapping φ_f for $f \in L^2(I)$ fixed, by

$$\begin{aligned} \varphi_f &: \mathbb{R}_+^* \rightarrow \mathbb{R}_+ \\ M &\mapsto \|P_{L^2(I)}g(f, M) - f\|_{L^2(I)}. \end{aligned}$$

Let (f_n, M_n) be a sequence in $L^2(I) \times \mathbb{R}_+^*$ such that f_n strongly converges to f in $L^2(I)$ and a sequence M_n converges to $M > 0$ in \mathbb{R}_+^* .

The stages of proof are the following ones:

1. φ_f is continuous in \mathbb{R}_+^* .
2. $P_{L^2(I)}g(f_n, M_n)$ strongly converges to $P_{L^2(I)}g(f, M)$ in $L^2(I)$.
3. $P_{L^2(I)}g(f_n, M_n)$ weakly converges to $P_{L^2(I)}g(f, M)$ in $H^2(\partial G)$.
4. Ψ is not continuous in \mathcal{D} .

3. IDENTIFICATION OF ROBIN TYPE COEFFICIENTS AND NUMERICAL RESULTS

We present in this paragraph an actual and original numerical method allowing to resolve the inverse problem of identification of Robin coefficients. Still in the thermal framework, once the flux and the temperature at the inaccessible boundary J , have been computed, we can evaluate the Fourier heat transfer coefficients q from eqn.(4).

The bounded extremal Problem 2.1 is then solved for $s = 0.6$ and for $f(z) = -2 + 1/z$, which provides us with the trace on I of the harmonic function $u = Re(f)$ together with that of its normal derivative, $\partial_n u = \partial_\theta Im(f)$ for the constrained $M_0 = \|f\|_{L^2(J)} = 2.91992407$. If q is associated to u and $\partial_n u$, and q_{comp} from $Re(g)$ and $\partial_n Re(g)$, we obtain the plots of Figure 1. Observe that the results are quite good, though the function to be recovered on J possesses a singularity in the plane (here at $(x, y) = (0, 0)$).

The results of continuation established in Theorem 2.1 indicate, that if one wants to find a best approximant g , it is necessary to choose a constraint sufficiently close to $M_0 = \|f\|_{L^2(J)}$. Figure 2 confirms this observation.

Now the constraint M_0 is an unknown of the problem, given that it is expressed itself according to f on J while measurements are made only on I . It is so useful to give a method allowing to determine this constraint.

3.1. Algorithm

Assume that we are given some nonnegative flux Φ such that $\Phi \not\equiv 0$ and let u_d be the measurements performed on $I \subset \mathbb{T}$.

1. Compute from the available data the restriction to I of the analytic function $f = u_d + i \int \Phi d\theta$;
2. Solve the **BEP** related to the data f on I , and a suitable constraint $M > 0$ and a reference function $f_I = 0$ defined on J . This gives $g = g(f|_I, M)$ on \overline{G} ;

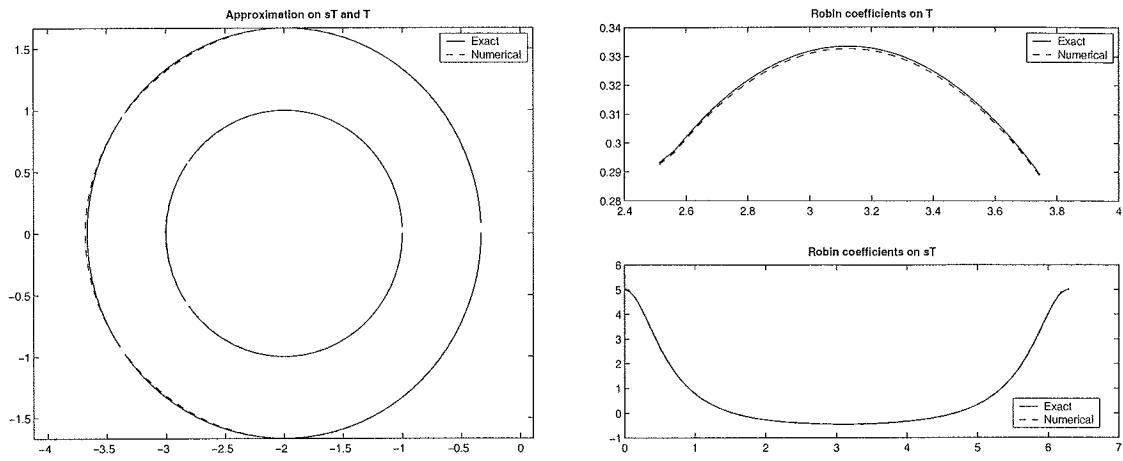


Figure 1. Plots of (g, f) and (q_{comp}, q) where $\theta_0 = 4\pi/5$.

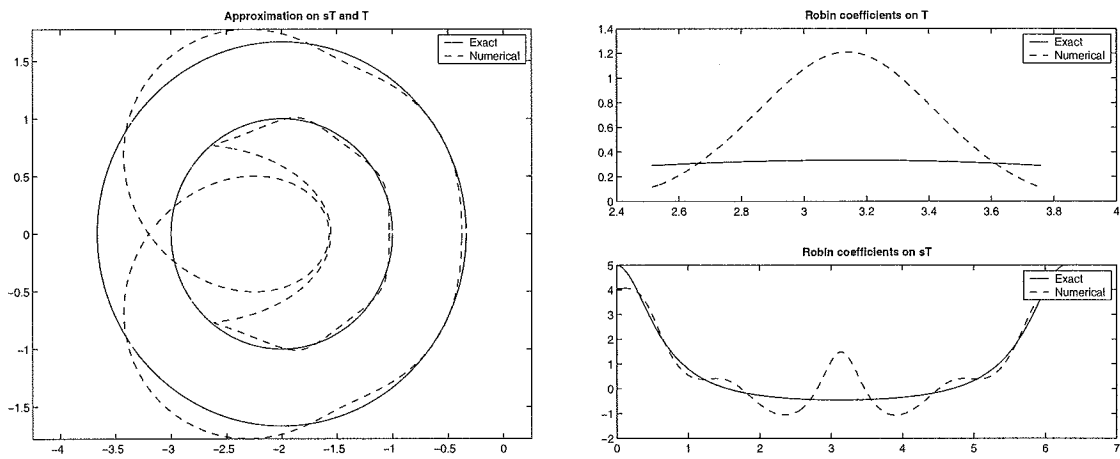


Figure 2. Plot of (g, f) and (q_{comp}, q) where $\theta_0 = 4\pi/5$ for $M = 2.5$.

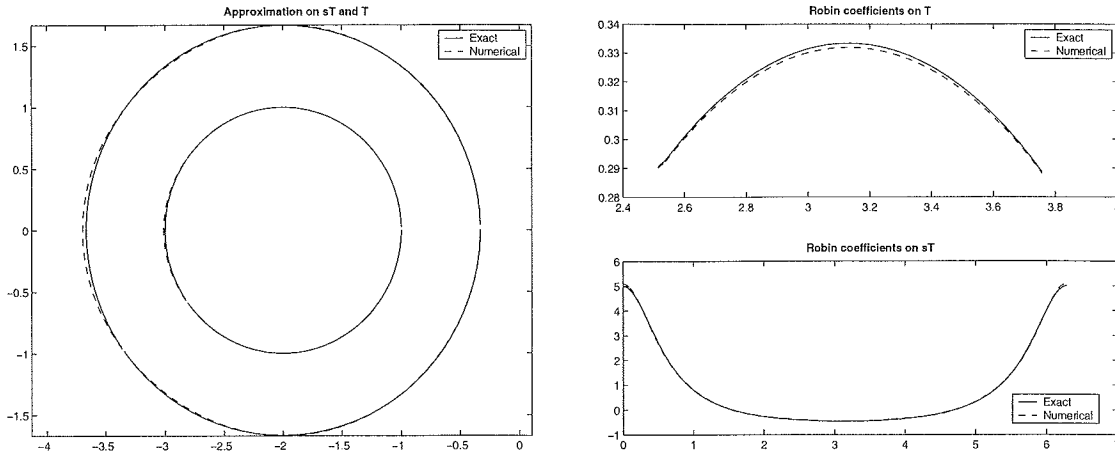


Figure 3. Plots of (g, f) and (q_{comp}, q) for $M_{numerical} = 2.924242424$ and $\theta_0 = 4\pi/5$.

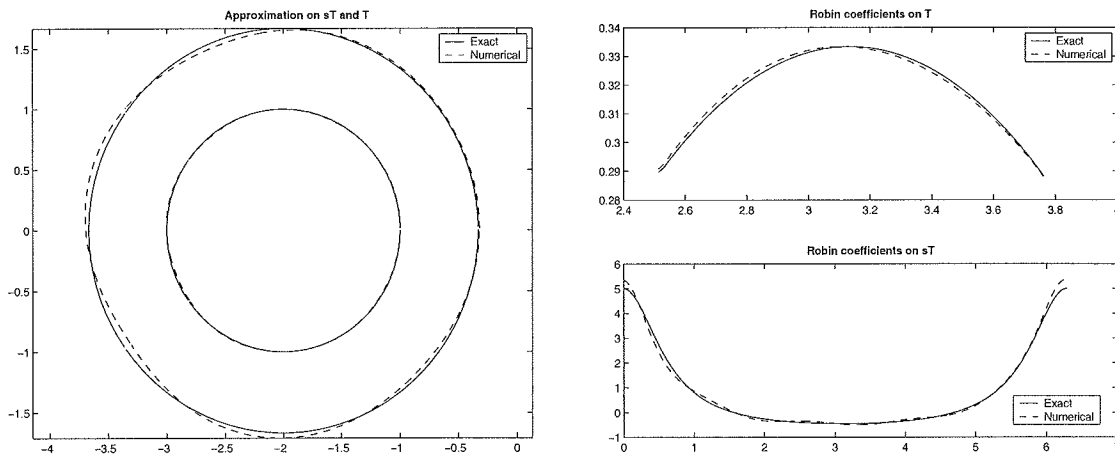


Figure 4. Plots of (g, f) and (q_{comp}, q) for $\theta_0 = 4\pi/5$ with noise 1%.

3. Compute

$$q = -\frac{\partial_{\theta} \text{Im}(g)}{\text{Re}(g)}.$$

Clearly the choice of M is crucial in this algorithm. There are several possible methods for determining a suitable value. One such is to use part of the available data to obtain an estimate for M as in [5]. An alternative is an iterative process based on a fixed point argument, allowing one to compute a solution corresponding to an appropriate value of M .

The second method described above gives an obtained value of $M = 2.924242424$ for the correct value of the constraint $M_0 = 2.91992407$.

Figures 3 and the following ones present the plots of (g, f) and (q_{comp}, q) for this value of M .

3.2. Noisy data

Noise is generated by a random variable whose uniform norm ranges from 1% to 10% of $\|f\|_{\infty}$. As expected from the robustness results of the above Subsection 3.1, the data extension process is less sensitive to noise than the Robin coefficient recovery method (Figures 4–6), although this latter is fairly robust.

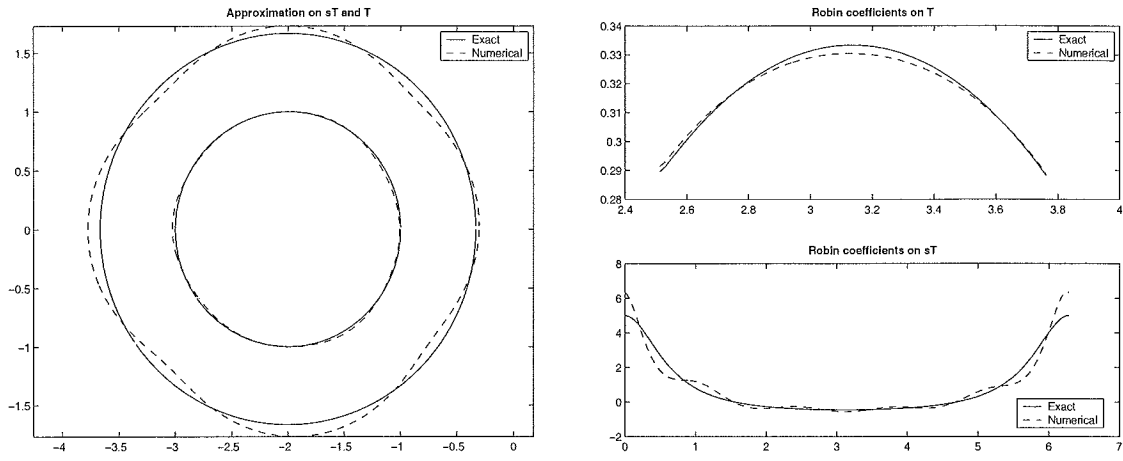


Figure 5. Plots of (g, f) and (q_{comp}, q) for $\theta_0 = 4\pi/5$ with noise 5%.

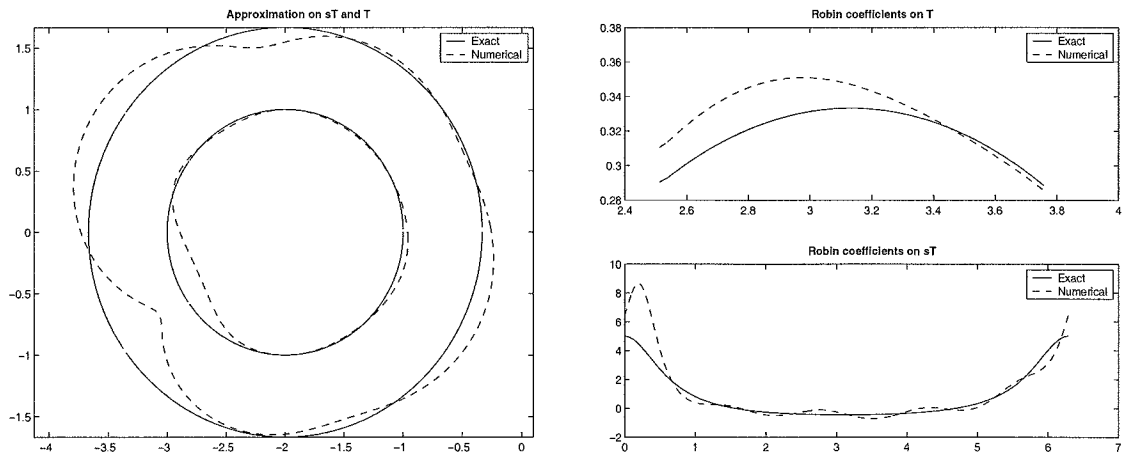


Figure 6. Plots of (g, f) and (q_{comp}, q) for $\theta_0 = 4\pi/5$ with noise 10%.

4. CONCLUSIONS

The method we have been presenting in this work first reads as a data completion one, solving the Cauchy problem for the Laplace equation. The problem framework, enriched with a cross validation procedure to control the instabilities inherent to such problems, turns out to provide an effective and robust method to build up the data extension.

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